1.

$$X(z) = \frac{3 + 5z^{-1} - z^{-2}}{(1 + 0.7z^{-1})(1 - 3z^{-1} + 2.25z^{-2})} = \frac{3z^3 + 5z^2 - z}{(z + 0.7)(z - 1.5)^2}$$

Use partial fraction expansion to determine coefficient values.

$$\frac{X(z)}{z} = \frac{A}{(z + 0.7)} + \frac{B}{(z - 1.5)} + \frac{C}{(z - 1.5)^2} + \frac{D}{z}$$

$$D = X(z)|_{z=0} = 0$$

$$C = \frac{X(z)(z - 1.5)^2}{z}\bigg|_{z=1.5} = \frac{265}{44} \approx 6.023$$

$$A = \frac{X(z)(z + 0.7)}{z}\bigg|_{z=-0.7} = \frac{-303}{484} \approx -0.626$$

Can solve for B by plugging in a value for z, or by taking $\frac{d}{dz}\frac{(z-1.5)^2 X(z)}{z}$

$$B = \frac{d}{dz}(z - 1.5)^2\ \frac{X(z)}{z}\bigg|_{z=1.5}$$

$$= \frac{d}{dz}\ \frac{3z^2 + 5z - 1}{(z + 0.7)}\bigg|_{z=1.5}$$

Use quotient rule to find derivative:

$$= \frac{(6z + 5)(z + 0.7) - (3z^2 + 5z - 1)}{(z + 0.7)^2}\bigg|_{z=1.5}$$

$$B = \frac{1755}{484} \approx 3.626$$

All together:

$$X(z) = -0.626\frac{z}{(z + 0.7)} + 3.626\frac{z}{(z - 1.5)} + 6.023\frac{z}{(z - 1.5)^2}$$

$$x(n) = \left[-0.626(-0.7)^n + 3.626(1.5)^n + \frac{6.023}{1.5}n(1.5)^n\right]u(n)$$

2. (a) First we must determine the magnitude of the poles of $H(z)$. This will let us know which poles must decrease in magnitude to make the system stable

```
% coefficients of b and a
b = [0.8581 4.2134 9.5802 9.5802 4.2134 0.8581];
a = [1 3.0937 5.5700 5.2578 2.0294 0.1642];

% find magnitude of poles
mag_a = abs(roots(a));
```

Which tells us that the magnitudes of the first two poles p_1 and p_2, where $p_1^* = p_2$, are greater than or equal to 1. In this case, $p_1 = p_2^* = -0.6979 + j1.3800$, $|p_1| = |p_2| = 1.5465$. Since there are 2 unstable poles our order, N, equals 2.

$$H_{un}(z) = \frac{1}{(z-p_1)(z-p_2)}$$

$$H_{un}(z) = z^{-N}H_{un}(z^{-1}) = \frac{z^{-2}}{(z^{-1}-p_1)(z^{-1}-p_2)}$$

With some rearranging, we get

$$H_{un}(z) = \frac{1}{|p_1|^2(z-\frac{1}{p_1})(z-\frac{1}{p_2})}$$

So that

$$H(z) = \frac{\frac{1}{|p_1|^2}B(z)}{(z-\frac{1}{p_1})(z-\frac{1}{p_2})\prod_{i=3}^{5}(z-p_i)}$$

In MATLAB, finding the new zeros and poles goes as follows

```
% scale down the first two poles, convert to coefficients
roots_a = roots(a);
roots_a(1) = 1 / roots_a(1);
roots_a(2) = 1 / roots_a(2);
a_new = poly(roots_a);

% scale down the entirety of the zero coefficients
b_new = b ./ (abs(mag_a(1))^2);
```

The transfer function of the new stable system results in

$$H(z) = \frac{B_{new}(z)}{A_{new}(z)}$$

$$B_{new}(z) = 0.3588 + 1.7618z^{-1} + 4.0058z^{-2} + 4.0058z^{-3} + 1.7618z^{-4} + 0.3588z^{-5}$$
$$A_{new}(z) = 1.0000 + 2.2815z^{-1} + 2.2176z^{-2} + 1.2505z^{-3} + 0.3781z^{-4} + 0.0287z^{-5}$$

(b) Pole/Zero plots for the new, stable system, compared to the old, unstable system.

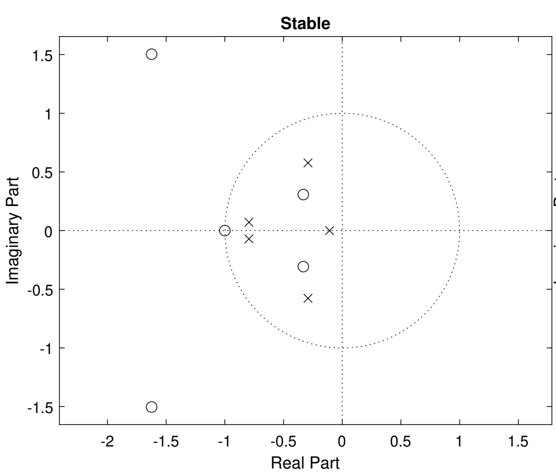

Figure 1: Stable Pole/Zero plot

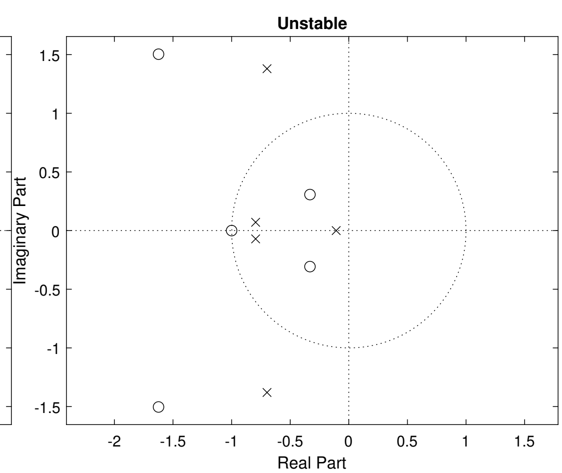

Figure 2: Unstable Pole/Zero plot

(c) Magnitude response for the new, stable system, compared to the old, unstable system from 0 to π. Response remains the same.

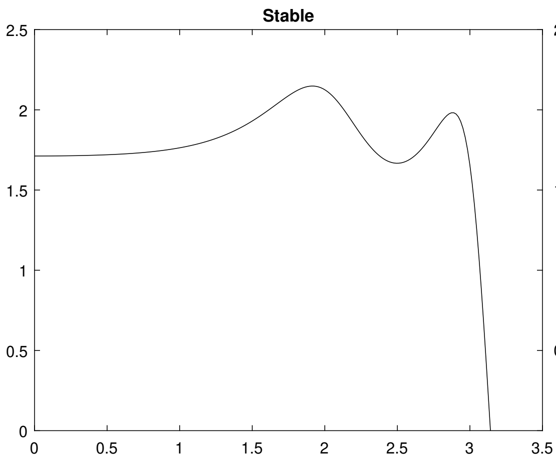

Figure 3: Stable magnitude response

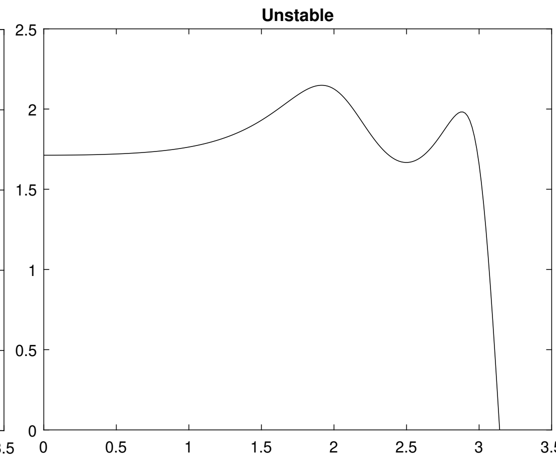

Figure 4: Unstable magnitude response

3. (a) The transfer function of the second order system is given by $H(z) = \frac{B(z)}{A(z)}$ with gain G of 2.4883 where zeros and poles are given by the following vectors in MATLAB:

```
% zeros of equal magnitude
q = [−0.7086+0.7056i, −0.7086−0.7056i, −0.4377+0.8991i, ...
    −0.4377−0.8991i, −0.4485+0.8938i, −0.4485−0.8938i, ...
    −0.5009+0.8655i, −0.5009−0.8655i, −1.0000];
% poles with decreasing magnitude
p = [−0.4305+0.9011i, −0.4305−0.9011i, −0.4183+0.8993i, ...
    −0.4183−0.8993i, −0.3583+0.8904i, −0.3583−0.8904i, ...
    −0.0915+0.7972i, −0.0915−0.7972i, 0.3854];
```

Complex conjugate pairs can be represented in the second order form as follows

$$(z-c)(z-c^*) = (z^2 - 2\mathfrak{Re}(c)z + |c|^2)$$

So that

$$B(z) = (z^2+1.4172z+1)(z^2+0.8754z+1)(z^2+0.8970z+1)(z^2+1.0018z+1)(z+1)$$

$$A(z) = \frac{(z^2+0.8610z+0.9973)(z^2+0.8366z+0.9837)(z^2+0.7166z+0.9212)}{(z^2+0.1830z+0.6439)(z-0.3854)}$$

Assuming the gain of the system refers to a steady state input, $\omega = 0$. Since $z(\omega)$ just equals $e^{j\omega}$, $z(0) = e^{j0} = 1$, so

$$G = H(1) = \frac{B(1)}{A(1)} \approx \frac{170.8931}{23.8756} = 7.1576$$

Need to lower the gain from 7.1576 to 2.4883, so

$$H(z) = 0.3476\frac{B(z)}{A(z)} \mid G = 2.4883$$

(b) The difference equation associated with the second order sections derived in part (a) can be derived from

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{L} b(k)z^{-k}}{1 + \sum_{k=1}^{L} a(k)z^{-k}}$$

so that

$$Y(z)\left[1 + \sum_{k=1}^{L} a(k)z^{-k}\right] = X(z)\left[\sum_{k=0}^{L} b(k)z^{-k}\right]$$

Coefficients of the denominator and numerator of $H(z)$ are found using the *poly* function in MATLAB. Then using the inverse Z-transform, we get

$$\begin{aligned}0.3476\,[x(n) + 5.1914x(n-1) + 14.6838x(n-2) + 27.4823x(n-3) + 37.0894x(n-4) +\\ 37.0893x(n-5) + 27.4821x(n-6) + 14.6835x(n-7) + 5.1913x(n-8) + x(n-9)] =\\ y(n) + 2.2118y(n-1) + 4.9238y(n-2) + 5.3376y(n-3) + 5.8012y(n-4) +\\ 3.0755y(n-5) + 1.8424y(n-6) - 0.0518y(n-7) - 0.0406y(n-8) - 0.2243y(n-9)\end{aligned}$$

And to get the difference equation, move all the time-shifted y components to the other side of the equality to solve for $y(n)$.

4. (a) Since the spectrum of $x_a(t)$ is represented by a rectangle function, $S_a(f)$ is the convolution of that rectangle with itself, so we know its a triangle function. We know $F_0 = 50$ Hz and

$$x_a(t) = 2F_0\,\text{sinc}(2F_0 t) \Longleftrightarrow X_a(f) = \Pi\left(\frac{f}{2F_0}\right)$$

$$s_a(t) = x_a^2(t) = 4F_0^2\,\text{sinc}^2(2F_0 t) \Longleftrightarrow S_a(f) = 2F_0\Lambda\left(\frac{f}{2F_0}\right)$$

Which is given by the plot below.

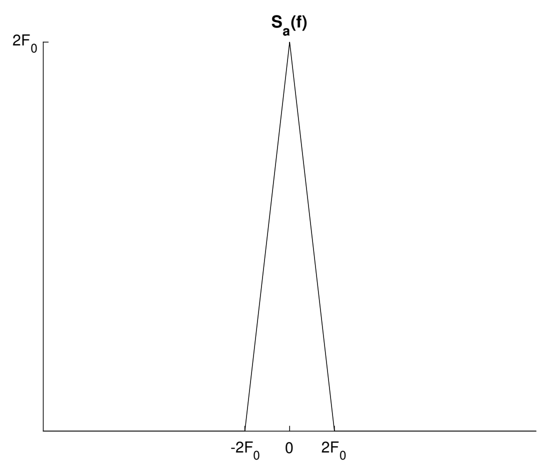

Figure 5: Plot of $S_a(f)$

After the ideal A/D converter, $s_a(t)$ is multiplied with an impulse train with form $\delta(t - nT_s)$ where n is the sample and T_s is the sampling period. This represents convolution in the frequency domain. We know the sampling frequency, $F_s = 250$ Hz, and

$$\delta(t - nT_s) \Longleftrightarrow F_s\delta(f - nF_s)$$

So

$$S(f) = S_a(f) \star F_s \sum_{n=-\infty}^{\infty} \delta(f - nF_s)$$

Which results in $S_a(f)$ scaled up by our sampling frequency, repeated at intervals of our sampling frequency. A plot of $S(f)$ is found below.

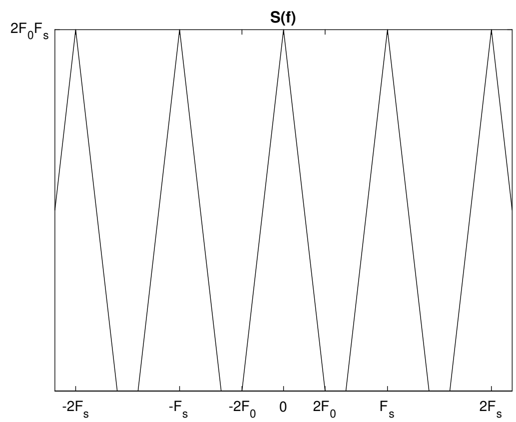

Figure 6: Plot of $S(f)$

Magnitude is $2F_0F_s = 25{,}000$. Visually, we can see that the Nyquist sampling rate, or the rate before aliasing occurs, is $2F_0$ Hz, or 100 Hz. After the signal $s(n)$ is reconstructed through the ideal D/A converter with the same rate F_s, the magnitude goes back to $2F_0$ and the repetitions stop for $|f| > \frac{F_s}{2}$. That cutoff frequency F_c is 125 Hz. Meaning only the range between $-F_c : F_c$ of $S(f)$ remains in $Y(f)$:

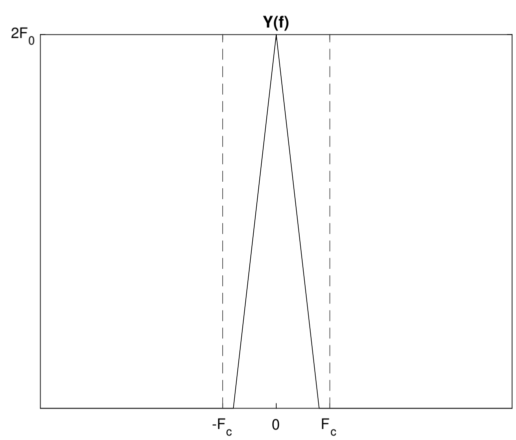

Figure 7: Plot of $Y(f)$

(b) Repeating part (a) but with $F_s = 150$ Hz. The plot of $S_a(f)$ remains the same. The plot of $S(f)$, however, looks like the following (still to scale from part (a)):

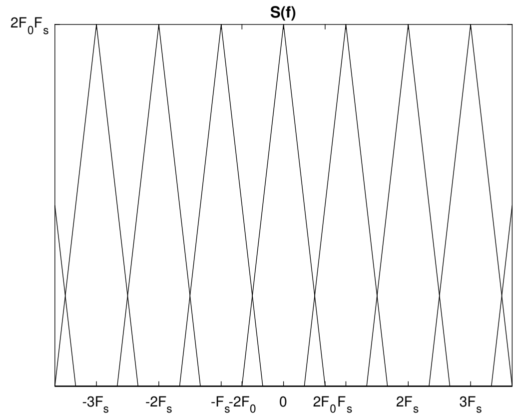

Figure 8: Plot of $S(f)$

Magnitude is $2F_0F_s = 15{,}000$. Visually, we can see that the Nyquist sampling rate, or the rate before aliasing occurs, is $2F_0$ Hz, or 100 Hz. In this case, aliasing occurs. $F_c = \frac{F_s}{2} = 75$ Hz. The range from $-F_c : F_c$ of $S(f)$ remains in $Y(f)$:

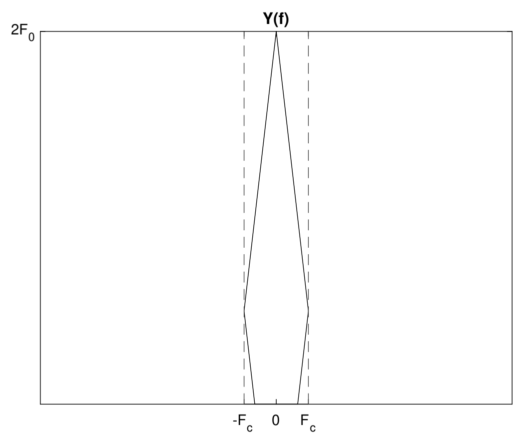

Figure 9: Plot of $Y(f)$

NOTE: *Assume that the overlapping portions in Figures 8 and 9 are added together; I can't seem to figure out how to do that in MATLAB.*

5. (a) $X_a(f)$ ranges from -5 kHz to 5 kHz, and sampled ($F_{s,A/D}$) at 10 kHz. Meaning, the spectrum of the sampled signal $X(f)$ is the following:

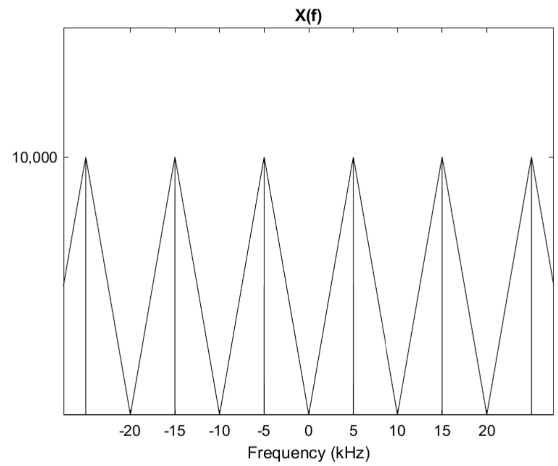

Figure 10: Plot of $X(f)$

No aliasing occurs. In this case, the sampling rate is exactly double the Nyquist frequency. An ideal discrete-time system low-pass filter with unity gain and F_c of $\omega = 2\pi/3$ is applied to $x(n)$ to produce $y(n)$. This is essentially two thirds of F_N (5 kHz), since our F_N determines the period before aliasing starts, shown by

$$(-F_N, F_N) \Longleftrightarrow (-\pi, \pi)$$

Therefore, going from $x(n) \rightarrow y(n)$ only the frequencies in range $-\frac{2F_N}{3}$ kHz : $\frac{2F_N}{3}$ kHz remain, which is where the vertical lines occur in Figure 11. With unity gain, the magnitude remains the same.

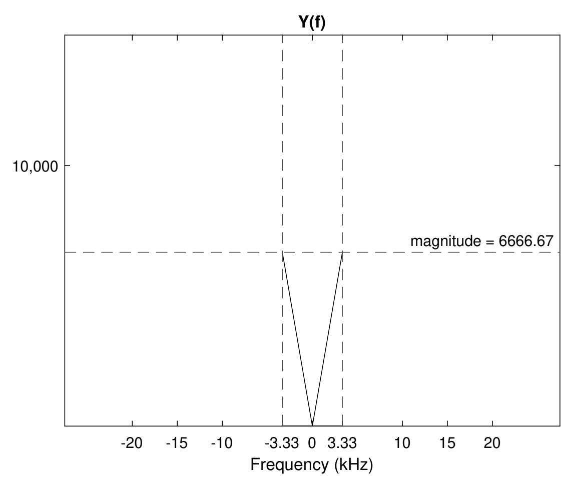

Figure 11: Plot of $Y(f)$

This gets passed through an ideal D/A filter to produce $y_a(t)$. The cutoff frequency occurs at $F_c = \frac{F_{s,D/A}}{2} = 5$ kHz. Therefore, the reconstructed signal looks like this

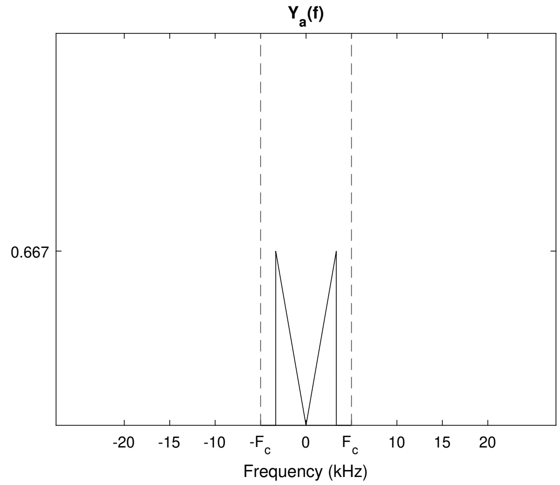

Figure 12: Plot of $Y_a(f)$

(b) Repeat but with $F_{s,A/D} = 15$ kHz and $F_{s,D/A} = 10$ kHz.

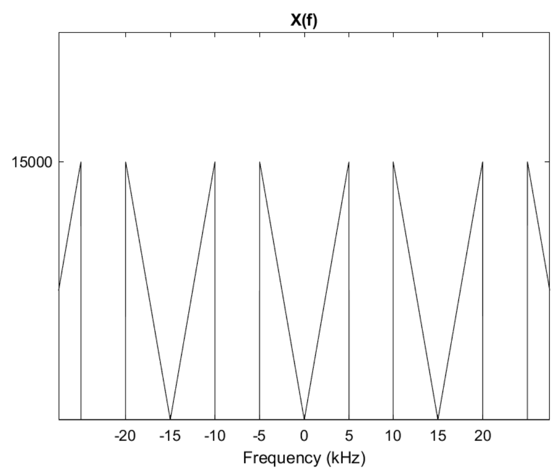

Figure 13: Plot of $X(f)$

No aliasing occurs. Nyquist frequency is 7.5 kHz. Going from $x(n) \to y(n)$ only the frequencies in range $-\frac{2F_N}{3}$ kHz : $\frac{2F_N}{3}$ kHz remain, which is where the vertical lines occur in Figure 14. With unity gain, the magnitude remains the same.

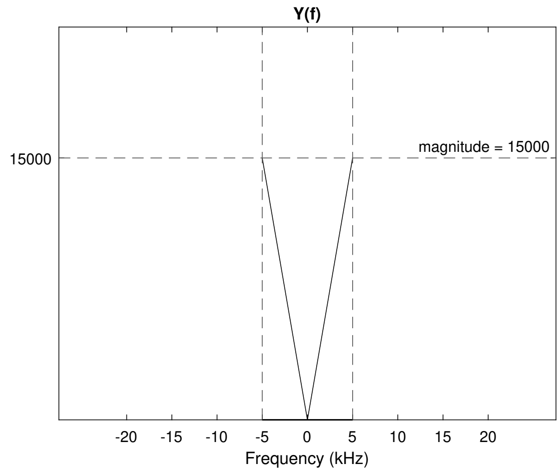

Figure 14: Plot of $Y(f)$

This gets passed through an ideal D/A filter to produce $y_a(t)$. The cutoff frequency occurs at $F_c = \frac{F_{s,D/A}}{2} = 5$ kHz. Therefore, the reconstructed signal looks like this

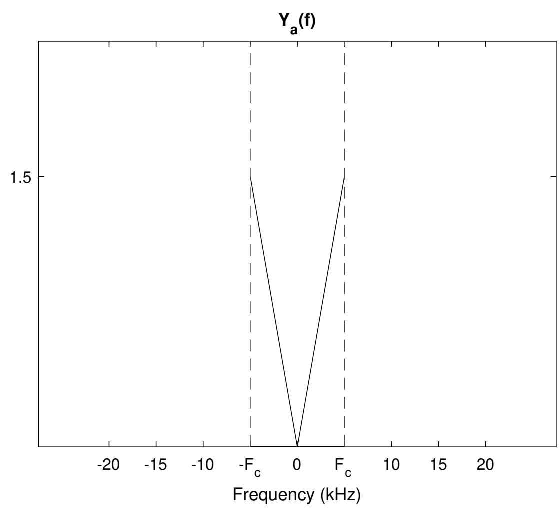

Figure 15: Plot of $Y_a(f)$

(c) Repeat but with $F_{s,A/D} = 8$ kHz and $F_{s,D/A} = 16$ kHz.

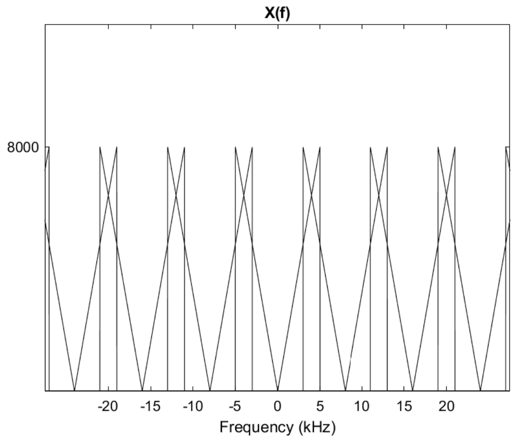

Figure 16: Plot of $X(f)$ with overlap

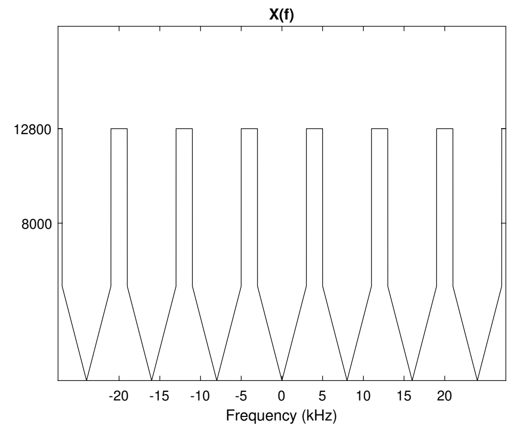

Figure 17: Plot of $X(f)$

Aliasing occurs; Nyquist frequency is 4 kHz. Going from $x(n) \rightarrow y(n)$ only the frequencies in range $-\frac{2F_N}{3}$ kHz : $\frac{2F_N}{3}$ kHz remain, which is where the vertical lines occur in Figure 18. With unity gain, the magnitude remains the same.

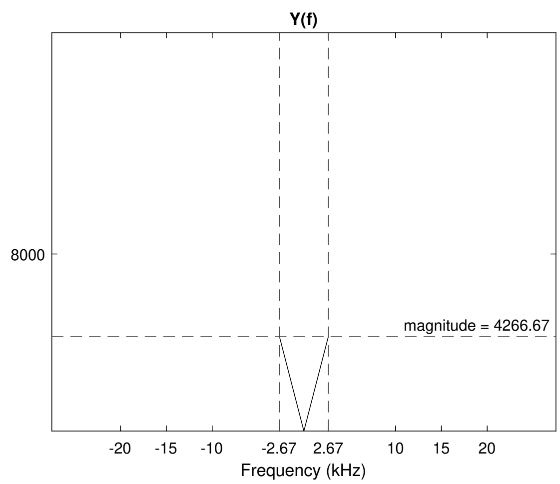

Figure 18: Plot of $Y(f)$

This gets passed through an ideal D/A filter to produce $y_a(t)$. The cutoff frequency occurs at $F_c = \frac{F_{s,D/A}}{2} = 8$ kHz. Therefore, the reconstructed signal looks like this

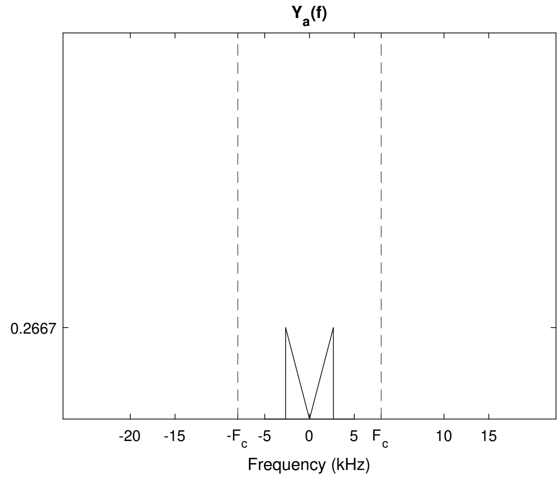

Figure 19: Plot of $Y_a(f)$